The spectrum from Lattice NRQCD

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I review recent results for heavy-heavy spectroscopy using Lattice NRQCD. The NRQCD collaboration reports that spin-independent splittings for the Υ are scaling for a sensible range of β values in the quenched approximation. Spin-dependent splittings are not, if the scale is set by spin-independent splittings. Results which include higher order spin-dependent relativistic and discretisation corrections show differences from previous (NRQCD collaboration) results without these. As expected, the differences are small for Υ but rather large for charmonium. New results from the SESAM collaboration for Υ spectroscopy on configurations with Wilson dynamical fermions show good agreement with previous results on HEMCGC configurations with staggered dynamical fermions.

1. New results

The new results using NRQCD which I will discuss are :

- SESAM collaboration results for the Υ spectrum on configurations with 2 flavours of dynamical Wilson quarks.
- Results from the NRQCD collaboration for the Υ spectrum on UKQCD quenched configurations at $\beta = 6.2$, giving a more detailed spectrum than previous results at this fine lattice spacing.
- Results for the Ψ and Υ spectrum which include additional relativistic and discretisation corrections in spin-dependent terms from Trottier, SESAM and UKQCD.
- Further B_c results on dynamical configurations and including relativistic c quarks with non-relativistic bs.

2. NRQCD

The splittings between radial and orbital excitations for systems made of heavy quarks are around 500 MeV, much less than the masses of the bound states. This implies that these are non-relativistic systems and a systematic expansion of the QCD Hamiltonian in powers of v^2 may be useful [1].

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The continuum action density, correct through $\mathcal{O}(M_Q v^4)$, is broken down according to

$$\mathcal{L}_{cont} = \psi^\dagger (D_t + H_0^{cont})\psi + \psi^\dagger \delta H^{cont} \psi \qquad (1)$$

H_0^{cont} and δH^{cont} are given explicitly in ref.[2]. On the lattice H_0 and the leading piece of δH, δH_1 are given by:

$$H_0 = -\frac{\Delta^{(2)}}{2M_Q^0} \qquad \text{and}$$

$$\begin{aligned}\delta H_1 \; =& -c_1 \frac{(\Delta^{(2)})^2}{8(M_Q^0)^3} + c_2 \frac{ig}{8(M_Q^0)^2}(\mathbf{\Delta}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{\Delta}) \\ & - c_3 \frac{g}{8(M_Q^0)^2}\sigma\cdot(\mathbf{\Delta}\times\mathbf{E} - \mathbf{E}\times\mathbf{\Delta}) \\ & - c_4 \frac{g}{2M_Q^0}\sigma\cdot\mathbf{B} + c_5 \frac{a^2\Delta^{(4)}}{24M_Q^0} - c_6 \frac{a(\Delta^{(2)})^2}{16n(M_Q^0)^2}\end{aligned}$$

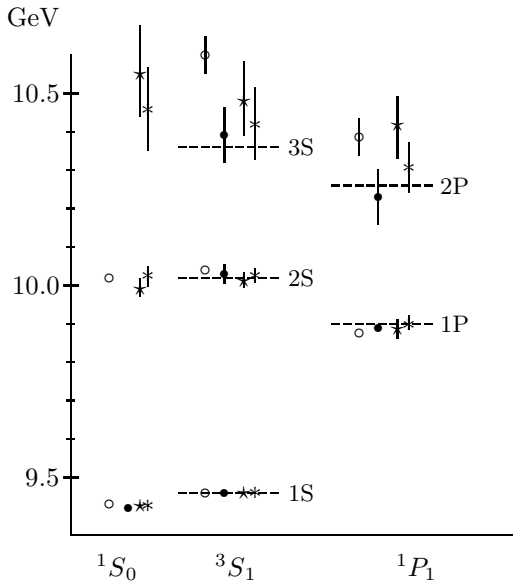
The last two terms in δH_1 come from finite lattice spacing corrections to the lattice Laplacian and lattice time derivative, of $\mathcal{O}(a^2 M_Q^3 v^4)$ and $\mathcal{O}(aM_Q^2 v^4)$ respectively. n is the stability parameter used in the evolution equation below.

The quark propagators are determined from evolution equations that specify the propagator value, for $t > 0$, in terms of the value on the previous timeslice;

$$G_1 = \left(1 - \frac{aH_0}{2n}\right)^n U_4^\dagger \left(1 - \frac{aH_0}{2n}\right)^n \delta_{x,0},$$

$$G_{t+1}=\left(1-\frac{aH_0}{2n}\right)^n U_4^\dagger \left(1-\frac{aH_0}{2n}\right)^n (1-a\delta H)G_t$$

The quark propagators are combined with smearing operators at source and sink to produce good overlap with different states. This is done in different ways by different groups, NRQCD and SESAM using Coulomb gauge wavefunction smearing and UKQCD, gauge-invariant blocking. All use multi-exponential fits to multiple correlation functions on large ensembles to obtain masses for radial excitations in s and p channels which can be compared to experiment.



--- : Experiment
$\circ$: NRQCD $(n_f = 0), \beta = 6.0$
$\bullet$: NRQCD $(n_f = 2, KS, am_q = 0.01), \beta = 5.6$
$\star$: SESAM $(n_f = 2, W, \kappa = 0.157), \beta = 5.6$
$*$: SESAM $(n_f = 2, W, \kappa = 0.1575), \beta = 5.6$

Figure 1. The spin independent spectrum for $b\overline{b}$

$$\begin{aligned}\delta H_2 \;=& -f_1\frac{g}{8(M_Q^0)^3}\{\Delta^{(2)},\sigma\cdot\mathbf{B}\}\\ &-f_2\frac{3g}{64(M_Q^0)^4}\{\Delta^{(2)},\sigma\cdot(\mathbf{\Delta}\times\mathbf{E}-\mathbf{E}\times\mathbf{\Delta})\}\\ &-f_3\frac{ig^2}{8(M_Q^0)^3}\sigma\cdot\mathbf{E}\times\mathbf{E}\end{aligned}$$

to be added to δH_1 above in the evolution equation. The action used is then correct through $\mathcal{O}(M_Q v^6)$ for *spin-dependent* terms. In addition these groups have included extra discretisation corrections for the spin-dependent terms in δH_1 at $\mathcal{O}(M_Q v^4)$, i.e. terms of $\mathcal{O}(a^2 M_Q^3 v^6)$. These involve [2] replacing the E and B fields with an improved version:

$$\begin{aligned}\tilde{F}_{\mu\nu} \;=&\; \frac{5}{3}F_{\mu\nu}-\frac{1}{6}[U_\mu(x)F_{\mu\nu}(x+\hat{\mu})U_\mu^\dagger(x)\\ +&\; U_\mu^\dagger(x-\hat{\mu})F_{\mu\nu}(x-\hat{\mu})U_\mu(x-\hat{\mu})\\ +&\; (\mu\leftrightarrow\nu)]\end{aligned}$$

and replacing the spatial derivative with an improved version (as was done for the leading spin-independent terms in δH_1) :

$$\tilde{\Delta}^{(2)} = \Delta^{(2)} - \frac{a^2}{12}\Delta^{(4)}. \qquad (2)$$

These corrections then appear as additional terms in δH_2. The results discussed here differ in how groups have treated E and B fields and corrections to derivatives in the spin-independent terms at $\mathcal{O}(M_Q v^4)$. This should not cause a big effect since these corrections are next-to-next-to-leading order in spin-independent splittings.

The c_i and f_i coefficients should be matched to full QCD either perturbatively or non-perturbatively. This has not been done by any of the groups. Instead they have relied upon tadpole-dominance arguments [3] to replace the gauge fields U_μ appearing in derivatives and E and B fields by U_μ/u_0. u_0 represents the effect of tadpoles in reducing the mean value of the link. Different values can be taken for u_0 and they will be compared below. After these modified gauge links are used in the action the c_i and f_i are set their tree-level values of 1. There

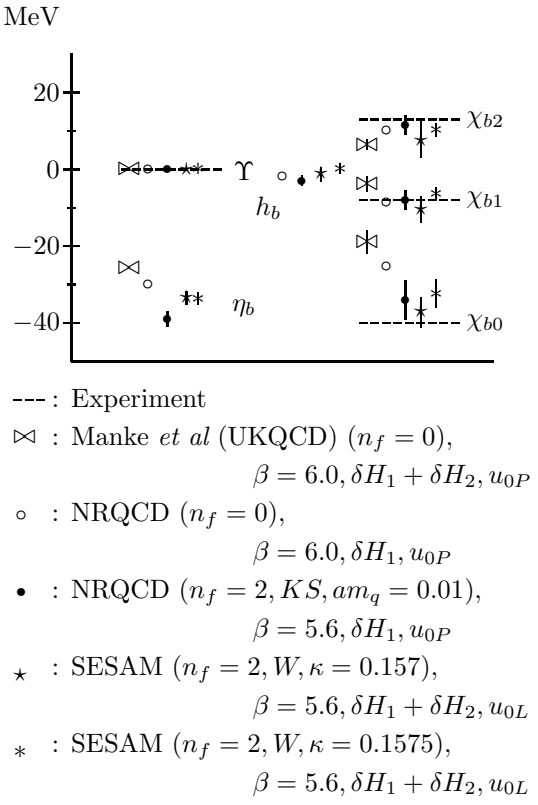


Figure 2. Υ fine structure

is evidence from perturbative calculations of the NRQCD self-energy diagram that the radiative corrections to this, at least for c_1 and c_5, are very small (less than 10%) for reasonable values of $M_Q a$ [4]. However, they could in principle be of the same size as the relativistic and discretisation corrections of δH_2.

There are two parameters then to fix in the lattice calculation: a and M_Q. For a, the 2S-1S or 1P-1S splitting should be used, where S and P are spin averages over s and p states. This has the advantage of being independent of quark mass experimentally in the b, c region. Because not all the states have been seen experimentally the $2^3S_1 - 1^3S_1$ and $1^3\overline{P} - 1^3S_1$ splittings are used in the $b\overline{b}$ sector, where the $^3\overline{P}$ is the spin

β, n_f	V.T	configs:	results:	Fig.
5.7,0	$12^3.24$	UKQCD	NRQCD	
6.0,0	$16^3.32$	Kogut *et al*	NRQCD	∘
6.2,0	$24^3.32$	UKQCD	NRQCD	
6.0,0	$16^3.48$	UKQCD	UKQCD	⋈
5.6,2	$16^3.32$	HEMCGC	NRQCD	•
5.6,2	$16^3.32$	SESAM	SESAM	⋆,∗

Table 1
Parameters for the ensembles used in the results discussed.

average of the $^3P_{0,1,2}$. To fix the quark masses the energy at finite momentum is calculated for one meson (say the 1^3S_1) and the denominator of the kinetic term in the dispersion relation is taken as the absolute mass of that meson in lattice units. The energy at zero momentum differs from this since the mass term was dropped from H_0. The difference between the energy at zero momentum and the mass can be compared to perturbative predictions and agrees well [5].

3. Results for Upsilon spectroscopy

The results to be discussed are tabulated below. Where a symbol is denoted in the last column that is the one used in Figures 1 and 2.

The HEMCGC configurations use Kogut-Susskind dynamical fermions and have two ensembles, one with $ma = 0.01$ and one with $ma = 0.025$. The NRQCD results on these configurations have been previously reported [6]. The matching quenched results at $\beta = 6.0$ represent a higher statistics study than that reported in ref. [5]. The SESAM configurations use Wilson dynamical fermions and have three ensembles. The two with lightest dynamical mass ($\kappa = 0.157$ and 0.1575) were used for the Υ spectrum calculations by Achim Spitz and Henning Hoeber. SESAM results on these configurations are reported elsewhere in this Proceedings [7].

Figure 1 shows the collected results for the spin-independent spectrum and Figure 2 the fine structure. The values of the lattice spacing used to convert to physical units have been taken as the

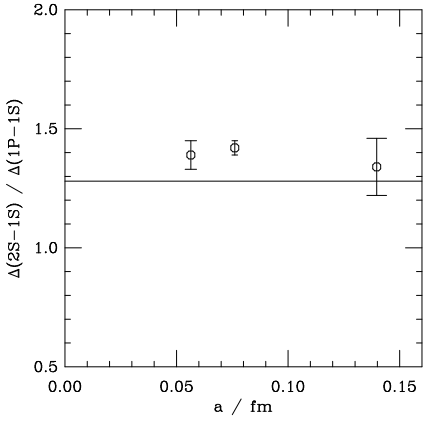


Figure 3. The ratio of splittings $2^3S_1 - 1^3S_1$ to $1^3\overline{P} - 1^3S_1$ in $b\overline{b}$ as a function of lattice spacing in fm at 3 different values of the lattice spacing on quenched configurations (NRQCD collaboration). The solid line represents the experimental result.

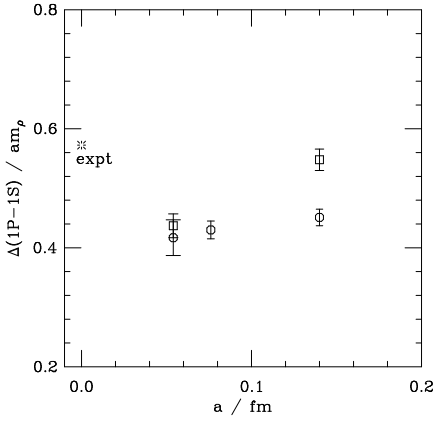


Figure 4. The ratio of the $1^3\overline{P} - 1^3S_1$ splitting in $b\overline{b}$ to the ρ mass as a function of lattice spacing in fm at 3 different values of the lattice spacing in the quenched approzimation. Υ results from the NRQCD collaboration. The circles use UKQCD results for the ρ mass, the squares GF11 results.

average of that from the 2S-1S and 1P-1S splittings. a^{-1} takes the value 2.4 GeV in all cases except for the heavier dynamical mass SESAM results in which it is 2.3 GeV. A bare b quark mass in lattice units of 1.71 was obtained by the NRQCD collaboration by tuning on the quenched $\beta = 6.0$ configurations. A value of 1.8 was used by them on the HEMCGC unquenched configurations but results from a kinetic mass analysis of the Υ [8] indicate that this was too large and 1.7 would have been better. The SESAM results use a bare b mass of 1.7.

Spin-independent spectrum - It is clear from the open circles in Figure 1 that the spin-independent spectrum on quenched configurations is not correct. We expect there to be errors because the coupling constant runs incorrectly between the scales appropriate to, say, the 1P and the 2S, so that it is not possible to fix an effective coupling which gives the correct answer for both states. Before comparing results at different values of n_f, however, we must first check that the results are scaling for a given value of n_f.

NRQCD is an effective theory reproducing the low energy behaviour of QCD but in which the ultra-violet cut-off plays a crucial rôle. It is therefore not possible to take a to zero within NRQCD. The c_i and f_i coefficients will start to diverge as powers of $1/M_Q a$ [4] and we will lose control of the NRQCD expansion. However, there is no need to take a to zero if we can demonstrate a independence of our results for a reasonable range of values of a. Provided $M_Q a > 0.6$ the c_i coefficients which have been calculated are perfectly well behaved [4] and we do not expect any problems. For the Υ system this corresponds to $\beta < 6.4$.

The NRQCD collaboration now has results at β values of 5.7, 6.0 and 6.2 in the quenched approximation (see table 1) [9]. Figure 3 shows the scaling of the ratio of the 2S-1S to 1P-1S splittings. There is no sign of significant scaling violations and the result clearly disagrees with experiment.

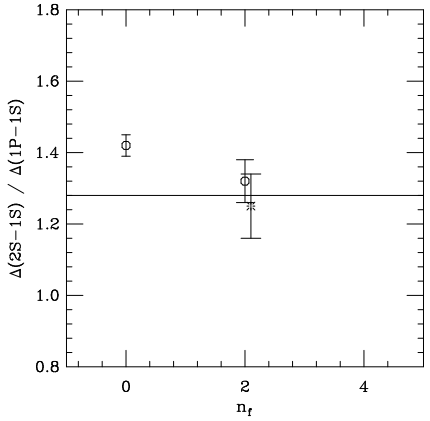


Figure 5. The ratio of splittings $2^3S_1 - 1^3S_1$ to $1^3\overline{P} - 1^3S_1$ in $b\overline{b}$ as a function of the number of dynamical flavours. NRQCD results are circles, SESAM results, a star. The solid line represents the experimental result.

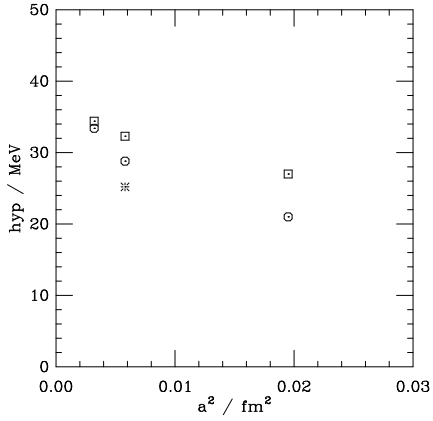


Figure 6. The $1^3S_1 - 1^1S_0$ splitting in MeV with the scale set by the $2^3S_1 - 1^3S_1$ splitting in $b\overline{b}$ as a function of the squared lattice spacing in fm^2 at 3 different values of the lattice spacing in the quenched approximation. NRQCD collaboration; the open circles use u_{0P} as in the simulation, the open squares are rescaled to give the results using u_{0L}. The star gives UKQCD results with a higher order action and u_{0P}.

These results use the action with δH_1 described above in which the leading $\mathcal{O}(M_Q v^2)$ terms are corrected for their lowest order discretisation errors. The remaining leading discretisation error is then expected to be the difference of terms of $\mathcal{O}(a^4 M_Q^5 v^6)$ between s and p states. At $\beta = 5.7$ the $\mathcal{O}(a^2)$ errors arising from the gluon field configurations generated with the unimproved Wilson plaquette action become significant. They can be corrected for perturbatively [6] and this has been done in Figure 3. It amounts to a 5% reduction in the 1P-1S splittings at $\beta = 5.7$, less than one σ in the ratio.

Figure 4 shows the ratio of the 1P-1S splitting in the Υ system to the UKQCD values of the ρ mass at the same three values of β. The UKQCD results shown are those for the tadpole-improved clover [10] light fermions. Good scaling is seen and very clear disagreement with experiment in the quenched approximation. For comparison I also show the ratio using the GF11 unimproved light hadron results [11] - clear violations of scaling are seen.

Having demonstrated scaling of the spin-independent spectrum at $n_f = 0$ (and therefore presumably at other values of n_f also), we can now study n_f dependence of the results. Figure 5 shows again the 2S-1S/1P-1S ratio plotted as a function of n_f. The results at $n_f = 2$ are in much closer agreement with experiment than at $n_f = 0$. The two results using different ensembles at $n_f = 2$ with different types of dynamical quarks are in good agreement with each other.

Since important momentum scales in Υ s and p states are around 1 GeV we might expect these splittings to 'see' 3 flavours of dynamical quarks. Extrapolating linearly through the NRQCD results at $n_f = 0$ and 2 does cross the experimental line at $n_f = 3$. Improved statistics at $n_f = 2$ would be useful to show definitively that this

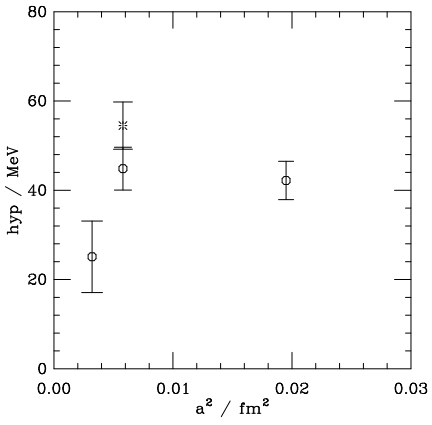


Figure 7. The $1^3S_1 - 1^1S_0$ splitting in MeV with the scale set by the $1^3P_2 - 1^3P_0$ splitting in $b\overline{b}$ as a function of the squared lattice spacing in fm^2 at 3 different values of the lattice spacing in the quenched approximation. Open circles, NRQCD collaboration; star UKQCD collaboration with a higher order action.

point gives incorrect answers.

The numbers plotted at $n_f = 2$ are for the lightest dynamical quark mass in both the NRQCD and SESAM cases. In principle the results should be extrapolated (possibly linearly [12]) in the light quark mass to the point

$$m_{dyn} = \frac{m_u + m_d + m_s}{3} \approx \frac{m_s}{3}. \quad (3)$$

However, no significant m_{dyn} dependence has been seen in either dynamical ensemble and so this has not been attempted.

Spin-dependent spectrum - Figure 2 compares the fine structure for the sets of results in Table 1, and includes details of the action used by each group. It is immediately obvious that the fine structure is much more sensitive to changes in the quark action. This is not surprising as the fine structure appears for the first time at an order one less in the relativistic expansion than the spin-independent spectrum. δH_1 used by the NRQCD collaboration contains only the leading order spin-dependent terms and so we expect changes at the 10% level ($v^2 \approx 0.1$) on using δH_2. It is also clear that unquenching has a big effect and will be necessary to get the right answers.

Comparable results with and without δH_2 are the $\bowtie$ from UKQCD [13] and the $\circ$ from NRQCD, both on quenched configurations at $\beta = 6.0$. A significant (15σ) effect is seen in the hyperfine splitting and the shift corresponds to a 10% effect. The shifts in the p fine structure may turn out to be somewhat larger (possibly as much as 20%) but currently they are not statistically significant. A comparison of the two collaborations results with only δH_1 shows agreement between the two different methods of smearing.

The $\bowtie$ points include both relativistic corrections (which are physical) and discretisation corrections (which are not). The scaling violations in the quenched NRQCD results can attempt to untangle these effects. Figure 6 shows a scaling plot of the hyperfine splitting divided by the 2S-1S splitting as a function of a^2 in fm^2. We expect violations of scaling at this order in the NRQCD results from errors in the B field in the $\sigma \cdot \mathbf{B}$ term which gives rise to this splitting. Figure 6 shows clear scaling violations with a slope μ given by $\approx$ 900 MeV, when we write [14]:

$$\left(\frac{hyp}{2S-1S}\right) = \left(\frac{hyp}{2S-1S}\right)_{a=0} \{1 - (\mu a)^2\}. \quad (4)$$

It is not surprising to find scaling violations of this size since the hyperfine splitting responds to much shorter distances than spin-independent splittings and is given by a single term in δH without the cancellation that occurs for derivative type discretisation terms in spin-independent splittings.

The hyperfine splitting is also sensitive to the value of the bare quark mass (like $1/M_Q a$) and errors in how well this is fixed will affect Figure 6.

In addition there is sensitivity to the value of u_0, because the B field contains 4 links. We expect the hyperfine splitting to vary as $1/u_0^6$, and this was borne out by work in ref. [5] where results with u_0 and without ($u_0 = 1$) were compared.

NRQCD and UKQCD results both use u_0 from the fourth root of the plaquette, denoted u_{0P} in Figure 2. Recent work by Trottier [15] has suggested that u_0 from the Landau gauge link, u_{0L}, might provide a better estimate of the radiative corrections to c_4. This might also be true for other c_i if u_{0L} captures tadpole effects more accurately. This value of u_{0L} is therefore used by the SESAM collaboration and has the effect of increasing the hyperfine splitting over the value that would be obtained with u_{0P}.

Figure 6 gives NRQCD values for the ratio of hyperfine to 2S-1S splittings that might be expected using u_{0L} by rescaling the results obtained with u_{0P} by the sixth power of the ratio of u_0 values. The u_{0L} results are somewhat flatter and this might indicate that some of the previous scaling violations arose from radiative corrections to c_4. Perturbative or non-perturbative calculations of various c_i will be needed to answer the question of which u_0 is better (if there is a single answer) and allow us to include radiative corrections in the leading coefficients in a consistent next-to-leading-order calculation of fine structure.

In n_f extrapolations of fine structure splittings it might not be true that $n_f = 3$ is the relevant physical point. $n_f = 4$ is somewhat more likely given the short distance nature of these quantities. In that case the extrapolations in n_f must be done in terms of other quantities for which $n_f = 4$ is also the physical point, and *not* the 1P-1S or 2S-1S splittings. Indeed, there will be no value of degenerate dynamical n_f for which we could get the right answer for the ratio of fine structure to spin-independent splittings. Instead, for example, we would have to extrapolate the ratio of the hyperfine splitting to one of the fine structure splittings. For this ratio better scaling is seen in the NRQCD results, as shown in Figure 7, although the results at $\beta = 6.2$ are still rather uncertain. So an extrapolation in n_f could then be done.

The results at different n_f are shown in Figure 8, where NRQCD results are compared at $n_f = 0$ and 2, and UKQCD results at $n_f = 0$ are compared to SESAM results at $n_f = 2$. The SESAM results are not strictly comparable with UKQCD since they use u_{0L}; if u_{0P} had been used

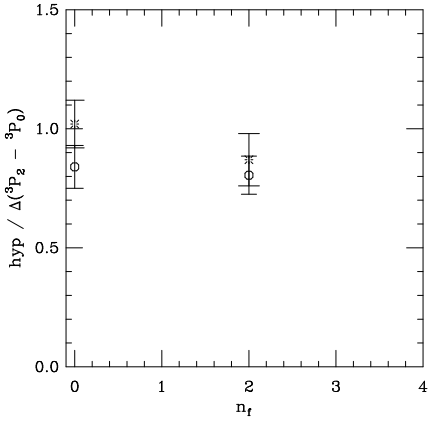


Figure 8. The ratio of the $1^3S_1 - 1^1S_0$ splitting to the $1^3P_2 - 1^3P_0$ splitting in $b\overline{b}$ as a function of n_f. Open circles, NRQCD collaboration; star, UKQCD collaboration at n_f =0, SESAM collaboration at $n_f = 2$. See text.

their result for the ratio plotted would be smaller by about 5%. Unfortunately the fine structure splittings are not very accurate and the resulting error in extrapolating the hyperfine splitting in this way is very large. At $n_f = 4$, the hyperfine splitting would be 40(10) MeV.

A comparison of the p fine structure to experiment from Figure 2 shows interesting features, albeit with large errors. The overall scale of the splittings given by ${}^3P_2 - {}^3P_0$ is obviously too small in the quenched approximation (and is reduced further when scaling violations are removed as for the hyperfine splitting). The unquenched results are much closer to experiment.

The ratio of splittings, $({}^3P_2 - {}^3P_1)/({}^3P_1 - {}^3P_0)$, is larger than experiment for the NRQCD results, 1.1(4) versus 0.66(2). That this is probably a discretisation error is borne out by the apparent improvement in this ratio in the UKQCD results. The ratio viewed in a potential model picture has differences of long and short range effects in it and it would be unlikely to be correct in the quenched

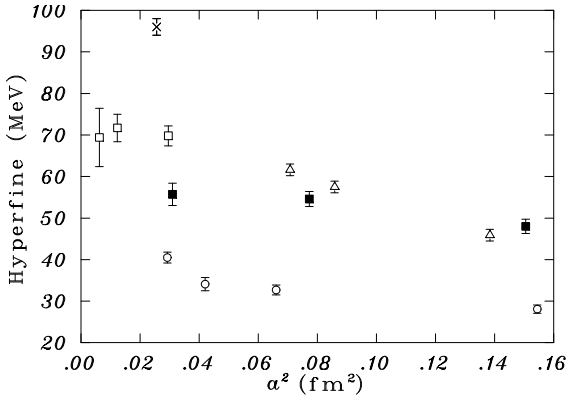


Figure 9. The hyperfine splitting for charmonium versus a^2 taken from ref. [15]. The next-to-leading order NRQCD results are given with u_{0P}, open circles, and with u_{0L}, filled boxes. The leading order NRQCD result [16] is marked with a cross. The relativistic heavy Wilson approach [17] gives the open boxes.

approximation, but is hard to determine accurately. The unquenched results in Figure 2 look encouragingly in agreement with experiment but a full study of scaling and n_f extrapolations must be done.

Obviously more work is required on the fine structure of the $b\overline{b}$ system and I believe that NRQCD will provide the most accurate results in this area. It will also be important to understand which quantities should be extrapolated to which values of n_f or, failing this, to make configurations with 'real world' dynamical quark content. Only then will we be able to get the correct answer for all the splittings without n_f extrapolations.

4. Results for Charmonium spectroscopy

The charmonium sector is a difficult one to simulate on the lattice, because it is neither very relativistic or very non-relativistic. For NRQCD to remain above $M_c a$ of around 0.6 requires $\beta < 5.85$ [16]. Each order of relativistic correction is only 30% smaller than the previous order since $v^2 \approx 0.3$. For the standard relativistic heavy Wilson approach [17] the opposite requirement is true, $\beta \geq 6.0$, because of problems with fixing the meson mass as a shift from the energy at zero momentum [18].

There has been a long standing issue [19] that results for charmonium hyperfine splittings disagreed between these two methods, with the NRQCD results giving a hyperfine splitting of around 100 MeV and the relativistic method around 70 MeV (experiment 116 MeV).

New work by Trottier in NRQCD [15] using the relativistic and discretisation corrections of δH_2 has shown that indeed the relativistic corrections from higher order terms in NRQCD are large. He finds the hyperfine splitting becomes smaller between the δH_1 results and the δH_2 results by a factor of 60%. This is certainly of $\mathcal{O}(30\%)$ so not in principle surprising but nevertheless disappointing to proponents of NRQCD.

A shift of the same size in spin-independent splittings would be a 15% effect (certainly of the same order as the naive expectation of 10%) since they are lower order in the non-relativistic expansion. This means a systematic error in a^{-1} determinations from NRQCD charmonium at this level if next-to-next-to-leading order spin-independent terms are not included. This is significantly less than the statistical error that can be achieved, but might nevertheless be acceptable in some applications.

It seems likely that future progress in the charmonium sector will use the heavy Wilson approach, possibly corrected for p^4 terms, as has been suggested [20].

5. Results for B_c spectroscopy

Recent experimental evidence for the B_c particle encourages lattice predictions for the spectrum of $b\overline{c}$ bound states. Early work [21] has used quenched configurations and NRQCD for both the b and the c quarks.

More recent results are collected in Figure 10. These include NRQCD collaboration results analysed by Martin Gorbahn on dynamical configurations from the MILC collaboration ($\beta = 5.415$,

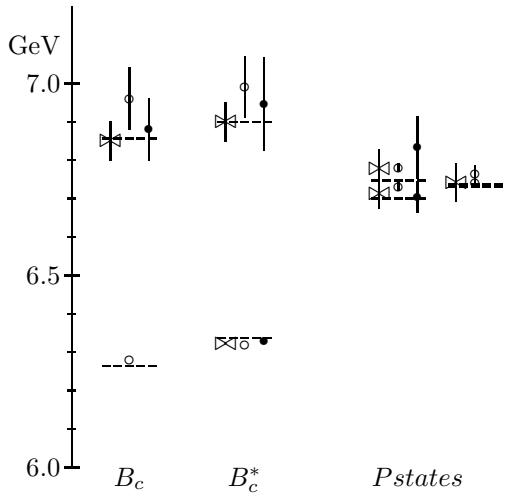


Figure 10. Lattice results for the spectrum of the B_c system. Open circles are results using NRQCD for both b and c quarks on quenched configurations at $\beta = 5.7$. Filled circles use unquenched MILC configurations. $\bowtie$ are results using NRQCD b quarks and heavy Wilson c quarks on quenched configurations at $\beta = 6.2$. Error bars represent statistical uncertainties only. Dashed lines show results from a recent potential model calculation [22]

$n_f = 2$, KS, ma =0.0125) and UKQCD results from Hugh Shanahan using NRQCD b and relativistic (tadpole-improved clover) c quarks on β = 6.2 quenched configurations. For comparison the older results [21] are shown and results from a potential model analysis [22]. There is no clear discrepancy with the potential model results as yet, despite the fact that the c quark is more relativistic in B_c than in charmonium.

One of the problems with a mixed system like the B_c is an ambiguity in what to take for the quark masses. This is quite significant in the quenched approximation because the value of a^{-1} depends on whether it is fixed from the Υ or Ψ system and the values of the bare quark masses have also been fixed separately within these systems. The splittings in Figure 10 will change somewhat if the quark masses were altered. To perform consistent extrapolations in n_f it will be necessary to use one particular splitting in, say, the Υ system to fix a^{-1} and then fix m_c using this a^{-1} in the Ψ system. This has not been done as yet.

One interesting feature of the B_c system is its similarity to heavy-light systems, allowing a test of some of the techniques that will be useful for the spectrum there. In particular the spin 1 p states will mix because of a lack of charge conjugation. It was possible for the quenched NRQCD results [21] to diagonalise the mixing matrix and pick out the physical 1^+ and $1^{+'}$ states. This will also need to be done in the B sector, but has not been possible as yet [23].

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